Journal of Statistical Physics, Vol. 78, Nos. 1/2, 1995

Turbulent Cascades

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Received November 4, 1993

Turbulent cascades at high Reynolds numbers are explained briefly in terms of multipliers and multiplier distributions. Two properties of the multipliers ensure the existence of power laws for locally averaged energy dissipation rate: (a) the existence of a multiplier probability density function that is independent of the level of the cascade, and (b) the statistical independence of multipliers at one level on those at previous levels. Under certain conditions described in the paper, the same two properties of multipliers guarantee that velocity increments over inertial-range separation distances also possess power laws. This work is specifically motivated by the need to understand the influence on scaling of the experimental observations that property (a) is true for turbulence, but property (b) is not; and additional motivation is the need to relate cascade models to intermittent vortex stretching (and folding). This effect has been modeled by allowing the multiplier distribution to depend on the magnitude of the local strain rate, and it is demonstrated that this rate-dependent model accounts for the statistical dependence observed in experiments. It is also shown that this model is consistent with the uncorrelated cascade models except for very weak singularity strengths (or for negative moments below a certain order), leading to the conclusion that, for all practical purposes, the uncorrelated level-independent multipliers abstract the essence of the breakdown process in turbulence.

KEY WORDS: Turbulence; cascades; turbulent energy transfer; multiscale interaction; nonlinear interaction; multipliers; multiplier distributions; statistical physics of turbulence.

1. INTRODUCTION

Turbulent motion at high Reynolds numbers is excited over a wide range of scales. Since turbulent energy is continually dissipated into heat by viscous action at small scales, the maintenance of a steady turbulent state

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0022-4715/95/0100-0311$07.50/0

requires a source of energy. This is usually provided by mechanical or thermomechanical stirring at some large scale or by the large-scale instability characteristic of the basic state of the flow. It then becomes necessary to imagine that the energy injected at the large scale, say L, somehow gets transmitted to the dissipative small scales. The process of energy transfer from large to small scales is a significant element of turbulence dynamics. Important contributions in this context have come from Richardson,[1] Kolmogorov,[2] Onsager,[3] von Weizsäcker,[4] Heisenberg,[5] and others.[2]

The first description of this energy transfer as both "local" and a "cascade" is due to Onsager.[3] If one utilizes Fourier representation of turbulence, the quadratic nonlinearity of Navier–Stokes equations shows that the energy transfer between any two wave numbers $\mathbf{k}$ and $\mathbf{k}'$ depends on the amplitudes of these wave numbers and their differences $\mathbf{k} \pm \mathbf{k}'$. If the wave numbers $\mathbf{k}$ and $\mathbf{k}'$ are both initially of the order $1/L$, the largest wave numbers participating in the interaction are typically of the order $2/L$. This reasoning applied to subsequent steps leads one to expect an energy transfer process in which wave numbers increase geometrically by a factor of the order 2 for each step of the process—this being the hallmark of "local" energy transfer in wavenumber space.[3]

In general usage, the term "cascade" has various meanings: (a) elements arranged in series so that each element is driven from the preceding element and in turn drives the succeeding one; (b) a series of vessels, from each of which there is a successive overflow to the next; (c) a waterfall over a slope, descending with ever-increasing speed; (d) a connected arrangement whose result is to produce a multiplicative effect of some sort that stops at a certain stage.[4] The appropriateness of the meanings (a) and (b) to turbulence is obvious from the previous paragraph. To see how (c) is appropriate, let us recall the empirically known fact[7] in high-Reynolds-number turbulence away from solid walls that the average rate of energy dissipation per unit mass follows the relation

$$\langle \varepsilon \rangle = Au^3/L \tag{1.1}$$

[2] Von Weizsäcker and Heisenberg collaborated on their turbulence work in the isolation of Farnhall in the summer of 1945. By the time the papers appeared, the authors had become aware through G. I. Taylor of Kolmogorov's as well as Onsager's work.

[3] While the Fourier representation of the turbulent field is natural for infinitely extended homogeneous systems, a less contrived description is often in terms of scales in real space. The important point is that the scale interaction in turbulence is local in a suitable wavelet representation,[6] thus implying some degree of simultaneous localization in both real and Fourier spaces.

[4] For example, in cascade showers which produce high-energy photons and electron–positron pairs when high-energy electrons pass through matter, the multiplicative effect stops when the energy of individual particles falls below a threshold value.

where $u^2/2$ represents the turbulent kinetic energy per unit mass and A is a constant of the order unity. Rewriting (1.1) as

$$\langle \varepsilon \rangle = Au^2/(L/u) \tag{1.2}$$

it is apparent that the energy injected[5] at the large scale L is lost by it in a time of the order L/u, which is its characteristic time scale. The energy transmission from any smaller scale $r = 2^{-n}L$, n being a positive integer greater than unity, to a scale half its size occurs at the fixed rate of $\langle \varepsilon \rangle$, so that the characteristic time for this process is of the order $\langle \varepsilon \rangle^{-1/3} r^{2/3} (L/u)\, 2^{-2n/3}$. It follows that, even if there is an infinity of steps in the cascade, the time taken for energy transmission across the entire wavenumber spectrum is of the order L/u. It is thus clear that the first step in the energy transfer essentially decides the overall speed of the process, with subsequent steps increasingly accelerated.

At all finite Reynolds numbers, the cascade terminates after a finite number of steps when scales of the order of the Kolmogorov scale η are reached. [The Kolmogorov scale is defined as $\eta = (\nu^3/\langle \varepsilon \rangle)^{1/4}$, ν being the kinematic viscosity of the fluid.] The energy density at all stages in the inertial range ($L^{-1} \ll k \ll \eta^{-1}$ or $L \gg r \gg \eta$, k and r being, respectively, the magnitude of the wavenumber vector and the representative scale of turbulent motion in real space) will be determined by the rate at which energy is being handed down from the first stage and eventually dissipated at the rate of $\langle \varepsilon \rangle$ per unit mass. The results obtained on the basis of this physical picture are well known and associated most often with Kolmogorov.(2) These results will not be described here; they can be found, for example, in the encyclopedic work of Monin and Yaglom.(8)

The physical picture just discussed has the important shortcoming that it neglects the intermittency of ε—by which one means its extreme variability in spatial distribution (see Fig. 1). Note that the large peaks in Fig. 1 are of the order of a few hundred times the mean, which shows that the fluctuations of ε from its average value cannot be ignored.[6] As the Reynolds number increases, the peaks in the distribution of ε become more and more singular, eventually becoming ill-defined over most of the space. Following a suggestion of Obukhov,(13) one therefore considers instead of ε its average ε_r over a volume of linear dimension r; it is believed that ε_r

[5] This interpretation makes the assumption—plausible but not obvious—that the kinetic energy of turbulence is of the order of the energy injected externally.

[6] The recognition of the importance of intermittency is usually traced(9) to a comment by Landau,(10) although Landau's comment literally referred to the nonuniversality that may arise from averaging ε over the nonuniversal large scales. For a good discussion of this point, see ref. 11. For a perspicacious review of the effects of intermittency, see ref. 12.

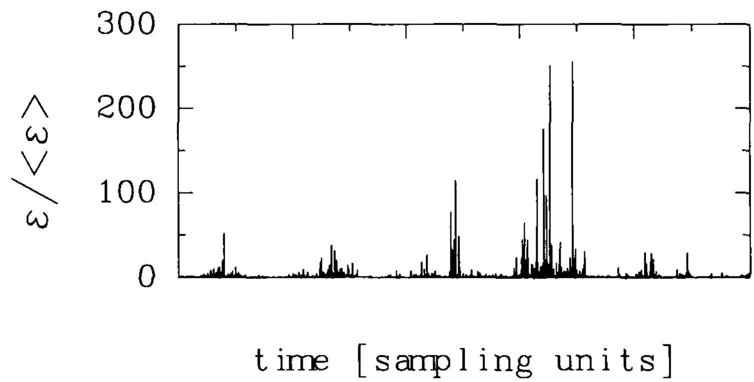

Fig. 1. A sample time trace of the square of the time derivative of streamwise velocity fluctuation in the atmospheric surface layer a few meters above the ground. By invoking Taylor's frozen flow hypothesis, the trace is interpreted as a one-dimensional spatial cut in the streamwise direction of one term of the energy dissipation rate, namely the square of the streamwise velocity derivative; for qualitative purposes, we shall treat the figure as representative of the distribution of the total energy dissipation rate as a function of a spatial coordinate. The abscissa spans a distance of about one integral scale. Note the tendency of the energy dissipation rate to cluster. The ordinate is normalized by its average over 100 integral scales; peaks which are several hundred times the global mean are not uncommon at this Reynolds number. The microscale Reynolds number based on the Taylor microscale and the root-mean-square velocity fluctuation is about 1500.

is well-behaved for all r in the inertial range. Note that $(r\varepsilon_r)^{1/3}/r$ represents the rate of strain experienced by a turbulent structure (loosely called a turbulent eddy) of size r. Intermittency of ε therefore implies that eddies of the same size are subject to vastly different rates of strain at different positions in space.

The broad task then is to determine the scaling properties of $r\varepsilon_r$, and to model turbulent cascades in a manner that is both realistic and consistent with this scaling. Section 2 gives a brief account of simple uncorrelated cascade models whose outcome is in agreement with experiment; the notion of multipliers and multiplier distributions is described, as is their relation to the scaling properties of $r\varepsilon_r$ and Δu_r. The goals of the paper are more explicitly spelled out toward the end of Section 2 by drawing attention to the somewhat paradoxical situation that these simple cascade models work better than can be expected reasonably. In particular, it is argued that uncorrelated cascade models do not take into account the complex process of eddy interaction and eddy splitting, which is the eventual source of interscale energy transfer in turbulence. This complex interaction is modeled in Section 3 by allowing the multiplier distribution to depend on the local rate of strain; the sense in which this rate-dependent cascade model is approximated by the uncorrelated cascade models is also discussed in this section. The paper concludes with a restatement of the principal results.

2. MULTIPLICATIVE PROCESSES, MULTIPLIER DISTRIBUTIONS, AND SIMPLE UNCORRELATED CASCADE MODELS

2.1. The Scaling of $r\epsilon_r$

Turbulence is three-dimensional in nature, but it is instructive to examine one-dimensional cuts of the three-dimensional field. In any case, this is useful pragmatism because experimental information, to which the outcome of the models is ultimately compared, is often in the form of one-dimensional cuts. Typical one-dimensional cuts miss certain types of structures (e.g., vortex filaments, such as those observed in the numerical simulations of ref. 14) or other rare events, and lead to nonstandard results such as negative dimensions;[15] they are, however, adequate for most other purposes. We proceed with the examination of one-dimensional cuts below; the generalization of the following phenemenology to three dimensions does not seem difficult in principle.

The underlying theme of cascade models is that the energy transfer occurring in the inertial range may be abstracted by a cascade process in which, typically, each eddy of size r subdivides into b pieces of size r/b in such a way that the energy flux per unit mass is redistributed unequally among b subeddies. The unequal split of the energy flux is modeled as the source of intermittency.[7] That being the case, the ith subeddy $(1 \leqslant i \leqslant b)$ will receive a fraction $M_i(r)$ of the energy flux carried by its parent eddy; the conservation of energy flux implies that $M_1(r) + \cdots + M_b(r) = 1$. (Note that, at high Reynolds numbers, the energy flux in the inertial range is strictly conserved in three dimensions. We are assuming that conservation holds also in one dimension. This introduces some formal difficulties which, however, lie outside the scope of this paper and will not be considered here.) In this manner, if we traced the history of all forefathers (up to the

[7] A special version of this model was developed in ref. 16; see also refs. 17 and 18. Other intermittency models such as the β-model of ref. 19, the random β-model of ref. 20, and the α-model of ref. 21 have been reviewed in ref. 22; see also ref. 23. A qualitatively different multiplicative process can be constructed by splitting an interval into a known number of unequal subintervals each of which inherits the same fraction of the energy flux.[24] This will not be pursued here even though it has some advantages of its own. In a different spirit, an intermittency model called the two-fluid model has also been developed recently.[25]

large scale L) of a given scale r, we would find that the energy flux $E(r)$ transmitted to a scale r is related to the flux $E(L)$ at scale L according to

$$E(r) = E(L)[M_1(L)\, M_2(L/b) \ldots M_n(L/b^{n-1})] \tag{2.1}$$

where $r/L = b^{-n}$. Since the energy flux at scale r can be written down as a product of the M_i, the latter will be called multipliers—first introduced by Novikov[26] and later discussed in greater detail by Mandelbrot.[27] As indicated in Section 1, the base b is expected to be about 2, and so we shall henceforth restrict our considerations to the binary case of $b = 2$.

The energy flux cascading down to smaller and smaller scales is converted into energy dissipation at the smallest scales, and so it is reasonable to suppose that the locally averaged energy dissipation $r\varepsilon_r$ can also be described analogously in terms of multipliers. Thus, the energy dissipation $r\varepsilon_r$ contained in an eddy (or scale or structure) of size r can be written in terms of the energy dissipation contained in an eddy of size L by

$$r\varepsilon_r = (L\langle\varepsilon\rangle) \prod_{i=1}^{n} M_i \tag{2.2}$$

where, for economy of notation, we have used the same symbol M_i in both (2.1) and (2.2); in any case, the symbol M_i will henceforth be used in the context of Eq. (2.2), and should not cause any confusion.[8] Note further that we have assumed that $\langle\varepsilon\rangle = \varepsilon_L$, which amounts to ignoring fluctuations in ε_L. The q-order moment of $r\varepsilon_r$ can be written as

$$\langle (r\varepsilon_r)^q \rangle = (L\langle\varepsilon\rangle)^q \left\langle \prod_{i=1}^{n} M_i^q \right\rangle \tag{2.3}$$

where the angular brackets $\langle\cdot\rangle$ indicate averages over multipliers associated with all scales of size r. By definition, the multipliers M_i in Eq. (2.2) can only range in magnitude between 0 and 1. Barring a deterministic energy transfer process (see Section 2.4), it is reasonable to suppose that the multipliers are random variables distributed on the unit interval, and to treat the averages in Eq. (2.3) as probability averages with respect to the joint probability density function (pdf) of the M_i.

Assume now that the multipliers at any given level are statistically independent of those in the previous level. *Assume* further that the

[8] Equation (2.2) says that the variability of ε_r is multiplicative, which justifies the usage of the term "cascade" in the sense (d) mentioned in Section 1. The multiplicative process stops when the Reynolds number of subeddies of size η is of the order of unity. Altogether, Onsager's choice of the word "cascade" seems to have been an inspired one.

multipliers at all levels in the inertial range have a unique pdf $P(M)$. We can then drop the suffix i in M_i in Eqs. (2.2) and (2.3) and rewrite Eq. (2.4) as

$$\langle (r\varepsilon_r)^q \rangle = (L\langle \varepsilon \rangle)^q \, (r/L)^{-\log_2 \langle M^q \rangle} \tag{2.4}$$

recalling that $r/L = 2^{-n}$. This shows that the moments of the energy dissipation contained in a piece (or an eddy) of size r vary as powers of r, with the scaling exponents given by the moments of the unique multiplier distribution.

As an aside, we may note that the multiplicative model can be written as an evolutionary equation for ε_r as

$$\frac{d\varepsilon_r}{dr} = \varepsilon_r(2M(r) - 1) \tag{2.5}$$

where $M(r)$ is the random multiplier at stage n given by $\log_2(L/r)$.

2.2. The Scaling of Velocity Increments

Consider the so-called velocity increments in the inertial range defined as

$$\Delta u_r = u(x+r) - u(x) \tag{2.6}$$

where u is the velocity component in the direction of the separation distance r. To relate the velocity increments to the locally averaged energy dissipation, recall the statement of the refined similarity hypothesis of Kolmogorov,[9] which, for present purposes, relates Δu_r to the energy dissipation $r\varepsilon_r$ over the interval r as

$$\Delta u_r = V(r\varepsilon_r)^{1/3} \tag{2.7}$$

where the stochastic variable V is independent of both r and $r\varepsilon_r$ for inertial range scales, and is hence universal. The validity of this hypothesis as a working approximation has been established experimentally[28] as well as by direct numerically simulations;[29] theoretical reasons for its validity have been explored in ref. 30. There are thus sufficient reasons to believe that Eq. (2.7) is correct (at least as a good approximation), and so we shall now use it with some confidence. If V is indeed independent of r and $r\varepsilon_r$, we obtain from Eq. (2.7) that

$$\langle (\Delta u_r)^n \rangle = \langle V^n \rangle (L\langle \varepsilon \rangle)^{n/3} \, (r/L)^{-\log_2 \langle M^{n/3} \rangle} \tag{2.8}$$

where n is an even integer.[9] Again, it is seen that moments of the multiplier distribution are central to the scaling properties of velocity increments in the inertial range.

Equation (2.8) suggests that the intermittency of ε_r is directly felt, via (2.7), by the velocity increments Δu_r as well. Experiments at all finite Reynolds numbers, including those characteristic of geophysical flows, show that velocity increments exhibit intermittency. The situation in the theoretical limit of infinite Reynolds numbers is obscure, however. There is indeed a body of literature that suggests that the intermittency of ε_r is not felt by the velocity field.[(31)] This latter conclusion is based on the analysis of structure functions in spectral space, using data generated numerically by the so-called shell models of turbulence, and their relevance needs to be assessed carefully.

2.3. Level-Independent Multiplier Distributions

The discussion so far shows that the existence of a unique level-invariant multiplier distribution, with multipliers at any level being statistically independent of those at all previous levels, guarantees the existence of power-law scaling of both $r\varepsilon_r$ and Δu_r. We shall now consider experimental data on the multiplier pdf and explore both level invariance and statistical independence. In order to determine $P(M)$, we may proceed as follows. Consider a long data string of ε distributed over an interval which is N integral scales in extent, N being some large integer. Divide each interval of size L into b equal-sized subintervals, and obtain the ratios of the measures (i.e., $r\varepsilon_r$) in each of the subintervals to that in the entire interval. These ratios are precisely the multipliers we want; they are clearly positive (since ε is positive definite) and lie between zero and unity. Subdivide each subinterval into b pieces as before, and repeat the procedure. At the nth level of this process, there will be Nb^n subintervals of size $r/L = b^{-n}$ and an equal number of multipliers. Construct the histogram of the multipliers at each level, and repeat the procedure until the smallest subinterval reached is of the order of the Kolmogorov scale η.

The pdfs $P(M)$ of the multipliers have been obtained in ref. 32 for different levels of subdivision; see also ref. 33. An important finding of ref. 32 is that the shape of the pdfs is independent of the scale r in the inertial range. This is shown in Fig. 2. (For very small r, this distribution

[9] $\langle (\Delta u_r)^q \rangle$ cannot be defined for nonintegers q. One can, however, empirically obtain the scaling exponents of $\langle |\Delta u_r|^q \rangle$ for any q. The formal relation between the scaling exponents of $\langle (\Delta u_r)^n \rangle$ and those of $\langle |\Delta u_r|^n \rangle$ is not clear for odd n.

is essentially uniform over the entire interval, whereas it approaches a delta function at 1/2 for r much larger than L.) This justifies our first assumption in Section 2.1 that there exists a unique level-independent pdf of multipliers in the inertial range.

Two observations will prove useful later. First, the scale-invariant pdf of Fig. 2 is fitted adequately by the β distribution (solid curve) given by

$$B_\beta(M)=\frac{\Gamma(2\beta)}{\Gamma(\beta)^2}M^{\beta-1}(1-M)^{\beta-1} \tag{2.9}$$

where Γ stands for the gamma function. The least-square value of β here is 3.2. We do not assign any theoretical significance to the β distribution, but view it as a convenient fit to the measured pdf. Second, in computing the multiplier pdf, if one randomly shuffled the dissipation among one-dimensional partitions of size R, say, the multiplier distributions are not scale invariant for any scale above R. The observed scale invariance thus incorporates to some degree the spatial coherence of the distribution of the measure.

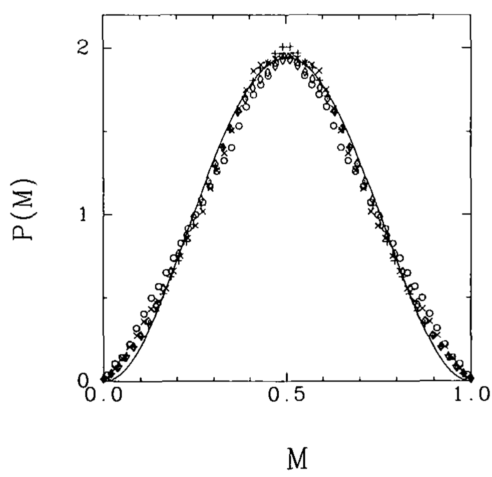

Fig. 2. Multiplier distributions for base 2. Different symbols correspond to different levels in the inertial range. Diamonds correspond to a starting level of 384η, where η is the Kolmogorov scale, pluses to 768η, crosses to 1536η, and circles to 3072η. The solid line is the β distribution, Eq. (2.9), with $\beta=3.2$.

2.4. Quasideterministic Models

We have shown elsewhere[34] that one can generate simple deterministic cascade models on the basis of the level-independent multiplier distributions just discussed. A few words about this procedure are appropriate here, especially because of the later need to employ it for similar purposes.

The cascade process with $b=2$ can be thought of physically as the breakdown of a structure (the parent eddy or scale) into two substructures at each level. Now, for the sake of simplicity, let us assume that one of the two subeddies at each level receives a fixed fraction p of the energy contained in the parent eddy; naturally, the other will receive $1-p$. This model is deterministic in the sense that the multipliers are fixed numbers independent of the level. (Strictly speaking, since there is no distinction between the left and right pieces, the model is only quasideterministic.) The multiplier distribution is written in terms of delta functions as

$$P_\delta(M) = \tfrac{1}{2}\,\delta(M-p) + \tfrac{1}{2}\,\delta(M-(1-p)) \tag{2.10}$$

If $p=1/2$, there is no intermittency and the situation corresponds to Onsager's and Kolmogorov's 1941 theory. To obtain intermittency, we should have a value of p different from $1/2$.

To determine a sensible value of p, it is natural to match the moments of $P_\delta(M)$ from Eq. (2.10) with those of the experimentally measured $P(M)$. For both distributions, the zeroth-order moment (normalization) and the first-order moment (mean value) coincide trivially, and are 1 and $1/2$, respectively. The first nontrivial condition is obtained by matching the second-order moment. This yields $p=0.7$. This is the so-called p-model of ref. 16, where it was obtained empirically by requiring a good fit to the multifractal data. The present analysis provides some rational basis for the model. It is a fortunate coincidence that the exact matching of the first three moments also matches high-order moments (at least up to order 7) quite well. This was shown in ref. 34 as a part of an overall scheme for constructing a hierarchy of quasideterministic models with increasing sophistication; the limitations of such a procedure were also noted there.

2.5. Conditional Distributions and Statistical Dependence

We now turn to the question of statistical independence, and obtain conditional pdfs of the multiplier M_i at level i given the multiplier M_{i-1} at level $i-1$. The scheme for doing so (Fig. 3) consists in computing $M_{i-1} = (r\varepsilon_r)_i/(r\varepsilon_r)_{i-1}$ (with no distinction between the left and right pieces), and forming the ensemble of intervals such that M_{i-1} falls within a narrow interval around a chosen value $M^{(-)}$, say. In this ensemble we compute the

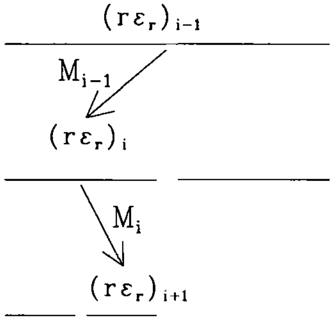

Fig. 3. The scheme for obtaining the multiplier distribution conditioned on the previous multiplier value.

normalized histograms of $M_i = (r\varepsilon_r)_{i+1}/(r\varepsilon_r)_i$ (where, again, no distinction is made between left and right pieces). The resulting histogram tends to the conditional pdf of M_i given $M^{(-)}$, $p(M_i \mid M^{(-)})$, when the number of samples in the ensemble tends to infinity and the size of the interval around $M^{(-)}$ tends to zero. The results are presented in Fig. 4; the dashed line corresponds to a narrow band around $M^{(-)} = 0.3$, the dotted line to a narrow band around $M^{(-)} = 0.7$, and the solid line to the average

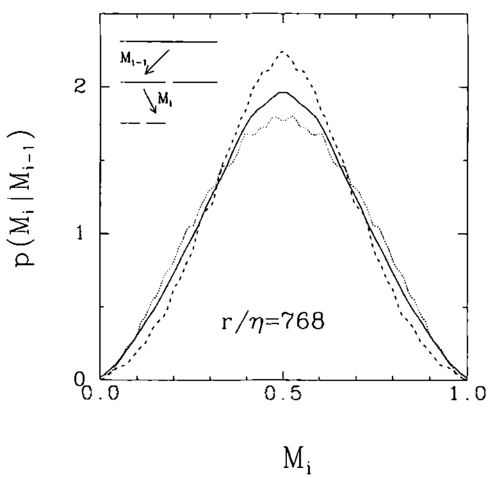

Fig. 4. The multiplier distributions conditioned on multiplier values at a previous level. The full line is the average of all symbols in Fig. 2 and thus represents the unconditional multiplier distribution. The dashed curve is for the multiplier M_i starting at a scale size of 768η given that the multiplier M_{i-1} at the previous level of twice the size is in the range [0.2–0.4]. The dotted curve is a similar distribution conditioned on the multiplier range [0.6–0.8].

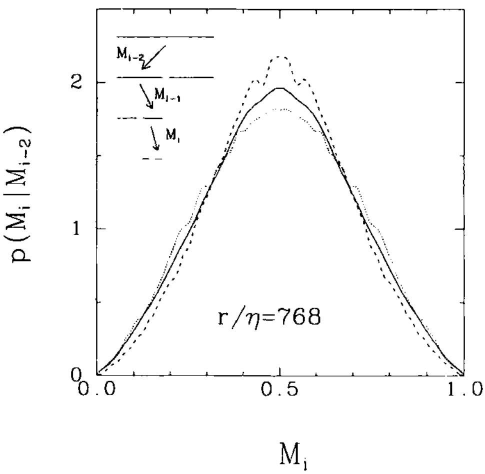

Fig. 5. The multiplier distributions conditioned on multiplier values from two previous levels. The full line is the same as in Fig. 4. The dashed and dotted lines represent conditional distributions of multipliers starting at a scale size of 768η. The conditioning ranges of the multipliers are the same as in Fig. 4.

unconditional pdf taken from Fig. 2. All pdfs are roughly scale invariant within the inertial range, but the multipliers at any given level do show a dependence on those at the previous level. This dependence persists even if one conditions on multipliers from two previous levels, as shown in Fig. 5. Evidently, the memory of the multipliers does not fade fast.

2.6. Recapitulation

We have seen that a sufficient condition for the existence of the power-law scaling of $r\varepsilon_r$ is that (a) the unconditional pdfs of the multipliers be scale invariant, and (b) the multipliers at one level be statistically independent of those at previous levels. It has been shown that (a) is surely true; just as surely, however, (b) does not hold. Yet, recall that one indeed observes at high Reynolds numbers a reasonably good scaling range (although it has not been demonstrated so far that the power-law scaling is a natural consequence of the governing equations). Further, if one assumes that a power-law scaling does occur, various independent experiments[22,35] yield exponents which are in good agreement with Eq. (2.4), an equation that takes this independence for granted. One is then led to conclude that, unless the experimentally observed scaling is forced (that is, power laws are fitted even where they do not really exist), the condition (b) is not necessary to observe scaling; at the least, the influence of the multiplier dependence on scaling should be benign in a sense yet to be understood. One of the goals of this work is to understand this aspect.

Even more basically, a realistic cascade model must at least allow a fragmentation process that is active in regions of large strain rate $(r\varepsilon_r)^{1/3}/r$ and inactive in regions of negligible rate of strain. The simple cascade models of Section 2 do not incorporate this feature in any way, yet they yield power-law exponents which agree with measurements. Is their performance merely fortuitous, or can it be understood as a rational approximation to a more complex physical model? We propose to address this question by first examining the dependence of multipliers on the magnitude of $r\varepsilon_r$, in particular the dependence of the distribution of multipliers M_i on the energy dissipation rate $(r\varepsilon_r)_i$. It will then be shown that this rate-dependent multiplier distribution contains the statistical dependence observed experimentally. Following this demonstration, we investigate the sense in which the simple cascade models of Section 2 are good approximations to the rate-dependent multiplier distribution, actually a simplified version of it.

3. MORE REALISTIC CASCADE MODELS

3.1. Rate-Dependent Multiplier Distributions

The conditional pdf of the multipliers M_i for some given $(r\varepsilon_r)_i$, $D(M_i \mid (r\varepsilon_r)_i)$, is shown in Fig. 6 for three values of $(r\varepsilon_r)_i$, with r fixed. The pdf is noticeably narrower for the smaller $(r\varepsilon_r)_i$ than for larger $(r\varepsilon_r)_i$, which simply means that there is less probable redistribution of energy flux among subeddies. If M is exactly 1/2, there is no redistribution of energy flux (which effectively means that there is no breakup at all). Regions of more intense dissipation correspond to zones of higher local Reynolds number [defined as $r(r\varepsilon_r)^{1/3}/\nu$] which can support a richer variety of instabilities leading to the breakup of eddies. In general, such a breakup leads to more drastic redistribution of the kinetic energy flux; values of M different from 1/2 will therefore occur more often than for smaller values of $r\varepsilon_r$. Note that, as for the unconditional pdf of Fig. 2, the pdfs of Fig. 6 are fitted well by the β distribution with varying values of β. The smooth curves in Fig. 6 are least-square β distribution fits to the measured pdfs.

In Fig. 6 we have examined the dependence of the multiplier distribution on the energy dissipation rate contained in boxes of fixed size r; it turns out that the distribution depends on r as well. This has been examined, and the values of β for various multiplier distributions corresponding to several values of r and $r\varepsilon_r$ have been obtained. The net outcome of this exercise is

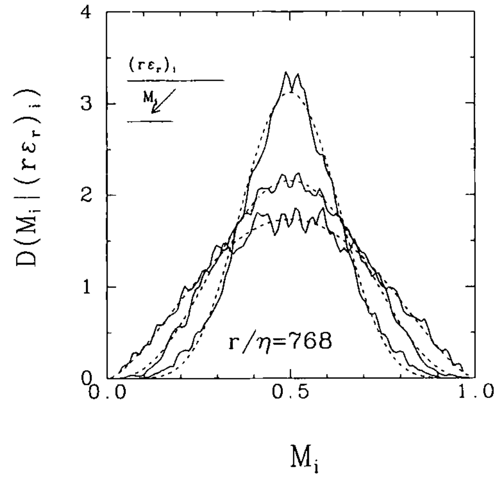

Fig. 6. The rate-dependent multiplier distributions corresponding to $r\varepsilon_r/\eta\langle\varepsilon\rangle$ taken from the windows [0–30], [60–90], and [300–700]. The lowest range corresponds to the most peaked distribution. For all three curves the multipliers are obtained by going from a scale of size 768η to half the size. Dashed curves in each case correspond to the β distribution with β values of 7.9 (for the most peaked distribution), 3.9, and 2.6 (for the least peaked distribution).

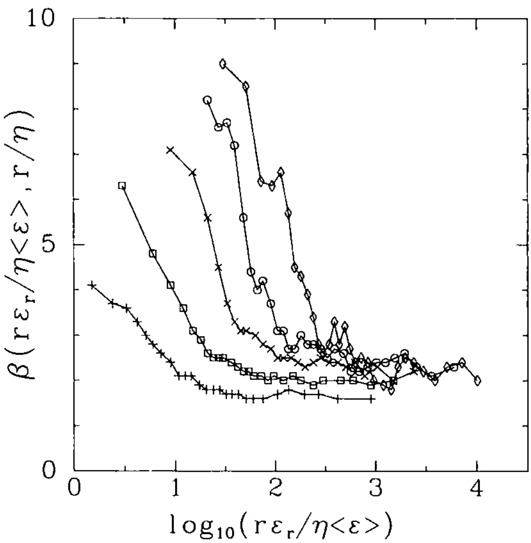

Fig. 7. The β values obtained from fits to the rate-dependent multiplier distributions, as a function of $r\varepsilon_r/\eta\langle\varepsilon\rangle$. Each curve corresponds to a different values of r. Plus: $r/\eta = 96$; square: $r/\eta = 192$; cross: $r/\eta = 384$; circle: $r/\eta = 768$; diamond: $r/\eta = 1536$.

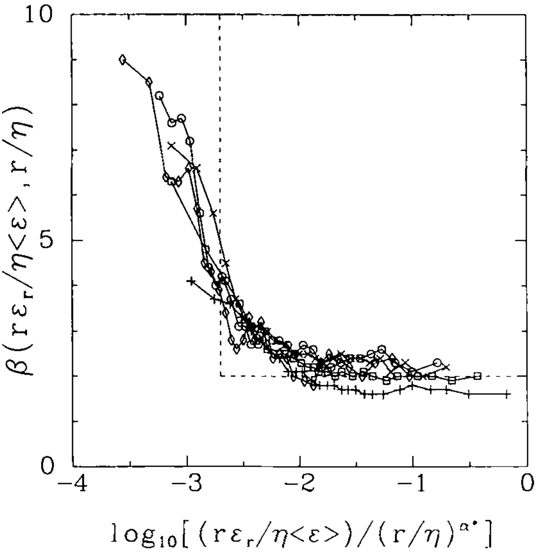

Fig. 8. The β values of Fig. 7 can all be collapsed roughly onto a common curve by normalizing $r\varepsilon_r/\eta\langle\varepsilon\rangle$ by $(r/\eta)^{\alpha^*}$, where α^* is 1.58. The dashed line is an idealization of this curve used to obtain the so-called correlated p-model described later in the text. The vertical line corresponds to $(r\varepsilon_r/\eta\langle\varepsilon\rangle)/(r/\eta)^{\alpha^*} = 0.002$.

the function $\beta(r/\eta, r\varepsilon_r/\eta\langle\varepsilon\rangle)$, shown in Fig. 7, where each symbol corresponds to different values of r. The main features of Fig. 7 are that for a fixed r, the parameter β decreases as $r\varepsilon_r$ increases, tending to an asymptotic value of approximately 2 for large $r\varepsilon_r$. All curves in Fig. 7 can be collapsed reasonably well (Fig. 8) if the parameter β is plotted versus

$$\frac{r\varepsilon_r}{\eta\langle\varepsilon\rangle}\Bigg/\left(\frac{r}{\eta}\right)^{\alpha^*}$$

with $\alpha^* \approx 1.58$. The theoretical significance of this numerical value of the exponent α^* is unclear at present, but we shall return to it momentarily. It appears clear, however, that the function $\beta(r/\eta, r\varepsilon_r/\eta\langle\varepsilon\rangle)$ is approximately of the form

$$\beta\left(\frac{r\varepsilon_r}{\eta\langle\varepsilon\rangle}\Bigg/\left(\frac{r}{\eta}\right)^{\alpha^*}\right)$$

Cascade models with multipliers depending on the magnitudes of the local energy flux or the local strain rate will be denoted henceforth as *rate-dependent* cascade models.

3.2. The Correlated p-Model

For later purposes, it is convenient to idealize the data in Fig. 8 by a β that is infinity when

$$\frac{r\varepsilon_r}{L\langle\varepsilon\rangle}\Big/\left(\frac{r}{L}\right)^{\alpha^*} \leqslant K$$

[which makes Eq. (2.8) degenerate to a delta function centered at $M = 0.5$], and by $\beta = 2$ when

$$\frac{r\varepsilon_r}{L\langle\varepsilon\rangle}\Big/\left(\frac{r}{L}\right)^{\alpha^*} > K$$

which makes the multiplier distribution parabolic. Further, in the spirit of the quasideterministic models described in Section 2.4, we shall represent the latter β distribution by an appropriate p-model, for which it is easy to show (by matching the first three moments as in Section 2.4) that $p = 0.72$. We thus arrive at the "correlated p-model" with the multiplier distribution given by

$$P\left(M \,\middle|\, \frac{r\varepsilon_r}{L\langle\varepsilon\rangle}, \frac{r}{L}\right) = \begin{cases} \delta(M-0.5) & \text{if } \dfrac{r\varepsilon_r}{L\langle\varepsilon\rangle} \leqslant K\left(\dfrac{r}{L}\right)^{\alpha^*} \\ \dfrac{1}{2}\,\delta(M-0.72)+\dfrac{1}{2}\,\delta(M-0.28) & \text{if } \dfrac{r\varepsilon_r}{L\langle\varepsilon\rangle} > K\left(\dfrac{r}{L}\right)^{\alpha^*} \end{cases} \tag{3.1}$$

with $\alpha^* = 1.58$. We shall use this model to establish the connection between the rate-dependent model on the one hand and uncorrelated cascade models on the other.

3.3. The Relation Between the Rate-Dependent and Uncorrelated Cascade Models

We now wish to establish, first, that the statistical dependence of multipliers at one level on those at previous levels is contained in the rate dependence of the cascade models. Second, we wish to show that rate dependence has no implications on all but negative moments below a certain order. Taken together, we will then have shown that the influence of statistical dependence from one level to another is negligible for all positive (and some negative) moments.

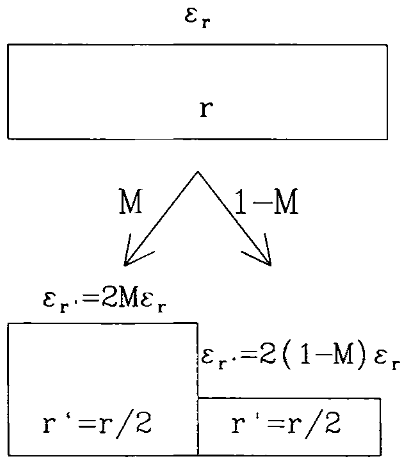

Fig. 9. The schematic of generating the dissipation signal using Eq. (3.2).

Toward the first objective, it is convenient to organize all the information on multiplier distributions in a single formula. The functional form

$$f\left(M \,\middle|\, \frac{r\varepsilon_r}{L\langle\varepsilon\rangle}, \frac{r}{L}\right) = GM(1-M)\exp\left(-\frac{K(r/L)^{\alpha^*}}{r\varepsilon_r/L\langle\varepsilon\rangle}\frac{1}{M(1-M)}\right) \tag{3.2}$$

where G is a normalization constant, adequately approximates the experimental pdfs for various values of r and $r\varepsilon_r$ (and, incidentally,[10] also the β distribution except in small neighborhoods of $M=0$ and 1). To generate a dissipation field using Eq. (3.2), one uses the generator of Fig. 9. Start with an interval of size L containing a total dissipation of $L\varepsilon_L$ and assume, as before, that $\varepsilon_L = \langle\varepsilon\rangle$. A multiplier M is now picked from the distribution of Eq. (3.2). This will lead to the first level of construction with two subintervals of size $r' = L/2$ and dissipation $(r'\varepsilon_{r'})_{\text{left}} = ML\varepsilon_L$ and $(r'\varepsilon_{r'})_{\text{right}} = (1-M)\,L\varepsilon_L$. In general, at a given level n where $r = L2^{-n}$, the values of r and the associated $r\varepsilon_r$ are inserted into Eq. (3.2) to pick a multiplier from it, which will then be used to determine the amount of dissipation on intervals at the following level. This process is repeated up to a maximum level N, where, as before, $\eta = L2^{-N}$. From the dissipation field so constructed, we can compute $p(M_i \mid M^{(-1)})$ numerically. We have done

[10] Incidental because the β distribution fit to (3.2) is not used anywhere in the text.

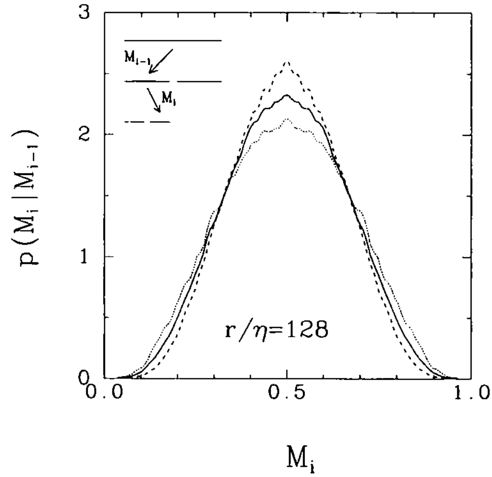

Fig. 10. The conditional multiplier distributions computed from the rate-dependent multiplier distributions; see text for details. The point to emphasize is that the rate-dependent multiplier distribution yields conditional multipler distributions which are similar to those in Fig. 4. The curves were obtained from a combination of 2^9 integral scales each with a ten-step cascade, i.e., $L/\eta = 2^{10}$.

this for $K = 1$ and $N = 10$ in Eq. (3.2). The resulting distributions, shown in Fig. 10, are similar to those shown in Fig. 4, thus confirming[11] that the dependence of M_i on M_{i-1} can be understood in terms of a somewhat more physical dependence of M_i on $(r\varepsilon_r)_i$.

For the second purpose mentioned earlier in this subsection, let us first note the following feature of the dynamics of the correlated p-model. According to Eq. (3.1), if at any particular level n (with $r = L2^{-n}$), the ratio

$$\frac{r\varepsilon_r}{L\langle\varepsilon\rangle} \Big/ \left(\frac{r}{L}\right)^{\alpha^*}$$

is less than K, or alternatively if $\alpha > \alpha^*$, the dissipation of an offspring at scale $r' = r/2$ is given by

$$\frac{r'\varepsilon_{r'}}{L\langle\varepsilon\rangle} = \left(\frac{r'}{L}\right)^{\alpha'} \tag{3.3}$$

[11] The multiplier distribution (3.2) shows a weak dependence on (r/L), at least for all finite number of steps in the cascade. However, this weak dependence does not negate the demonstration that *rate dependence* and the dependence of M_i on M_{i-1} are consistent with each other.

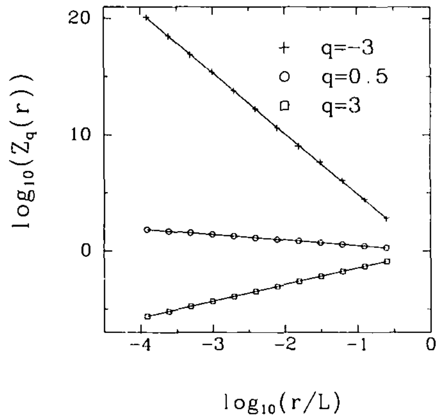

Fig. 11. The logarithm of the partition function defined in Eq. (3.4) plotted as a function of the logarithm of r/L, for the three values of parameter q displayed in the figure. The dissipation field was computed from the correlated p-model. The slopes of these curves yield the parameter τ. The principal point to emphasize is that the correlated p-model yields excellent scaling.

taking $K = O(1)$, say. Now,

$$\frac{r'\varepsilon'_r}{L\langle\varepsilon\rangle} = \frac{1}{2}\frac{r\varepsilon_r}{L\langle\varepsilon\rangle} = \frac{1}{2}\left(\frac{r}{L}\right)^\alpha = 2^{\alpha-1}\left(\frac{r'}{L}\right)^\alpha$$

which must equal $(r'/L)^{\alpha'}$ from Eq. (3.3). This is possible only if $\alpha' < \alpha$. This means that the correlated p-model tends to prevent local exponents α in $r\varepsilon_r/L\langle\varepsilon\rangle \sim (r/L)^\alpha$ from getting larger than α^*. In the multifractal language, this correlated cascade process imposes an upper bound α^* on the exponent α. Noting that larger α means weaker singularities, one might imagine that the standard uncorrelated p-model will work for larger singularity strengths.

To better establish that this is indeed the case, we computed from the dissipation field given by the correlated p-model[12] the partition function

$$Z_q(r) = \sum_{i=1}^{N} (r\varepsilon_r)_i^q \tag{3.4}$$

where the sum extends over all the N possible intervals of size r. This partition function scales with r as $Z_q(r) \sim (r/L)^{\tau(q)}$, as can be seen in Fig. 11 for $q = -3$, 0,5, and 3. The slopes of these plots (in the log–log representation) yield the value of the scaling exponents $\tau(q)$. The importance of $\tau(q)$ is that[27] we can compute from it the scaling of the moments of the dissipation $\mu(q) = \tau(q) + 1$, the generalized dimensions[27,36] $D_q = \tau(q)/(q-1)$, and

[12] The calculations use a multiplier value of 0.7 instead of 0.72 in the correlated p-model, but this makes no substantial difference to the conclusions.

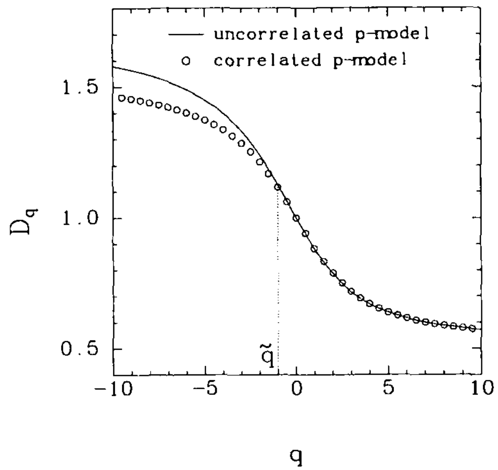

Fig. 12. The generalized dimensions D_q as a function of q, obtained from plots such as Fig. 11, compared with those obtained for the p-model of ref. 16. It is clear that there is no difference between the correlated and uncorrelated models for $q > \tilde{q}$. Differences show up for more negative q.

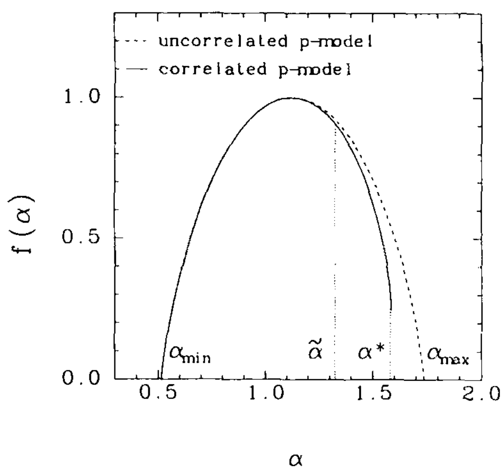

Fig. 13. The information in Fig. 12 has been converted via Legendre transforms to the $f(\alpha)$ curve. Again, it is seen that the correlated and uncorrelated models are nearly indistinguishable for all $\alpha < \tilde{\alpha}$.

the spectrum of singularities, or the $f(\alpha)$ curve,[37] via the Legendre transform. Figure 12 shows the generalized dimensions D_q^c for the correlated p-model (circles) and D_q^u corresponding to the uncorrelated p-model (solid curve). It can be seen that D_q^c and D_q^u coincide for $q > \tilde{q}$, with $\tilde{q} \approx -1$. That is, the scaling exponents of the moments of the dissipation should be approximately equal for the correlated and the uncorrelated processes except for moments of order $q < \tilde{q}$.

Alternatively, these same effects can be studied by using the spectrum of singularities $f(\alpha)$ shown in Fig. 13. The dashed curve corresponds to the uncorrelated p-model, and spans a range from $\alpha_{\min}$ [equal to $\log_2(1/0.7)$] to $\alpha_{\max}$ [equal to $\log_2(1/0.3)$]. The solid curve comes from the correlated p-model, spanning an α range from the same $\alpha_{\min}$ to α^*. As was the case with the generalized dimensions, the $f(\alpha)$ results from the correlated and uncorrelated p-models coincide for α between $\alpha_{\min}$ and $\tilde{\alpha} = \partial\tau/\partial q|_{\tilde{q}}$. (This last relation obtains from the fact that α and τ are related via Legendre transform.) However, for α larger than $\tilde{\alpha}$, differences arise between the two $f(\alpha)$ curves.

It must be noted that the possibility of a truncated $f(\alpha)$ curve was recognized in ref. 38.

To summarize, the observed statistical dependence of the multipliers has little effect on the scaling exponents of $\langle (r\varepsilon_r)^q \rangle$ for all $q > -1$.

4. CONCLUSIONS

We have considered energy cascades in high-Reynolds-number turbulence, and discussed the concept of multipliers and multiplier distributions. To guarantee the existence of power laws for locally averaged energy dissipation rate, it is sufficient for the multipliers to have two properties: (a) the level invariance of their probability density function, and (b) statistical independence of multipliers at one level of those at previous levels. Upon invoking the refined similarity hypothesis of Kolmogorov, these two conditions also are sufficient for the moments of the inertial-range velocity increments to vary as known powers of the separation distance. Closer inspection reveals that while the multiplier distributions are indeed level independent to a good approximation, multipliers at any given level show a significant dependence on those from previous levels. The issue then is to understand the manner in which this statistical dependence affects the scaling of $r\varepsilon_r$ and Δu_r. Finally, it was thought useful to make a connection between cascade models on the one hand and the phenomenon of intermittent strain rate (related to vortex stretching and folding) in turbulence.

We have approximated this last effect by means of a rate-dependent cascade model which allow the multipliers to depend on the magnitude of the local strain rate, and shown that it contains the statistical dependence of multipliers at one level on those at the previous level. We have abstracted the essence of this rate-dependent model by the so-called correlated cascade model, and shown that the latter coincides with the uncorrelated cascade model except for very weak singularity strengths. Thus, the statistical dependence observed in the experiment does not have any effect on scaling properties of turbulence except for negative moments below a certain order. At any rate, it can be said that statistical independence between multipliers is sufficient but not necessary for the power-law scaling observed in turbulence for all positive powers.[13]

In summary, we have explored the physical content of simple uncorrelated cascade models and examined their mathematical and physical consistency. We believe that this work enables us to appreciate the sense in which such models approximate what is evidently a more complex reality, so that they could be used with more confidence about their validity and better awareness of their limitations.

ACKNOWLEDGMENTS

We thank the Yale University Archives for permission to examine Lars Onsager's unpublished documents (available on microfilm). Among these documents is a section entitled "Turbulent Cascades," which inspired the title of this paper. The documents show that Onsager returned to a study of turbulence sometime after moving from New Haven to Miami. However, their cursory reading does not particularly reveal that he obtained any major result.

The paper was read carefully by S. Grossmann, who offered several penetrating comments (also, upon request, some historical remarks on refs. 4 and 5), R. H. Kraichnan, M. Nelkin, and L. M. Smith. To them and to G. K. Batchelor, who confirmed that the phrase "turbulent cascades" first appeared in Onsager's work, we owe our thanks. The work was supported financially by a grant from AFOSR.

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[13] The recent results of Novikov(39) are not inconsistent with this conclusion.

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